ON A TUMOR MONITORING EQUATION

BY

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Dedication

I wish to dedicate this thesis to my parents,
my brothers, my wife, my kinds and my friends.

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All praise and glory to Almighty Allah who gave me the courage and patience to carry out this work. Peace and Blessings of Allah be upon His last prophet Muhammad (Sallallah-Alaihe-Wasallam) and his family, who guided us to the right path. First and the foremost acknowledgement is due to the Department of Mathematics and Statistics at King Fahd University of Petroleum and Minerals, which provided a genial research environment and a wonderful working place. I am also indebted to KFUPM for supporting my research work.

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Thesis Abstract

Name:	Bader Al-Shareef
Title:	On a Tumor Monitoring Model
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A mathematical model of brain tumor in the presence of a treatment therapy is considered. The killing rate induced by the treatment which is assumed to be variable in time and space. Assuming the tumor to be of spherical shape, a model of resulting diffusion equation is formulated in spherical coordinates. The Lie-symmetry analysis is carried out to find the symmetry algebra which is employed to obtain some exact solutions. A non-linear killing rate is also studied.

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ملخص الرسالة

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لقد تم دراسة نموذج رياضي لتورم المخ في وجود العلاج الطبيعي. يعتبر معدل القتل الناجم عن العلاج متغير في الزمان والمكان. وعلى افتراض أن الورم ذو شكل كروي, فقد تم صياغة هذا النموذج الرياضي على شكل معادلة الانتشار في الإحداثيات الكروية.

لقد تم دراسة التحليل التماثلي الخاص بلي لإيجاد مجموعات جزئية من محاور التماثل ومن ثم استخدامها لإيجاد بعض الحلول التامة. وفي النهاية قمنا أيضاً بدراسة النسبة القاتلة اللاخطية.

ماجستير في العلوم

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CHAPTER 1

INTRODUCTION

The growth of tumor in brain tissues has been modeled as a reaction diffusion process. The cells are known to spread very fast and thus kill the healthy ones by depriving them of nutrients and space. To be able to destroy the tumor treatment must move faster than the rate of spread of tumor. One of the models that has been used is the so-called Burgess equation [6]. This is based a diffusion process coupled with the therapy dependent killing rate. The Burgess equation is given by

$$\frac{\partial n(r,t)}{\partial t} = D\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial n(r,t)}{\partial r}\right) + Pn(r,t) - kn(r,t), \tag{1.1}$$

where $n(r,t)$, the concentration of tumor cells, depends only on radial distance r. D is the diffusion coefficients which measures the invasiveness of tumor cells, P is the proliferation rate and k is the killing rate. We can write equation (1.1) in a parameter free form as

$$\frac{\partial n}{\partial t} = \frac{\partial^2 n}{\partial r^2} + \frac{2}{r}\frac{\partial n}{\partial r} + n, \tag{1.2}$$

Equation (1.2) can be reduced to a simpler form by putting $u(t,x) = r\ \ n(t,r), x = r$ as

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u, \tag{1.3}$$

which is a linear homogeneous heat equation which can be solved by separation of variables. However, a more interesting case arises when the equation (1.1) is written as

$$\frac{\partial^2 u}{\partial x^2} - K(x,t)u - \frac{\partial u}{\partial t} = 0, \tag{1.4}$$

where $K(x,t) = -(P-k)$ and $u(t,x) = rn(r,t)$ is used. Equation (1.4) has been investigated by Moyo and Leach by performing symmetry analysis of the equation for some specific forms of $k(x,t)$. We present some exact solutions of equation (1.4) using symmetries of (1.4) and obtain some new solutions. A more interesting case of the Burgess equation arises when $K(x,t) = -(P-K)$ depends upon the concentration u of the tumor cells. This model is based upon the studies indicating that the killing rate is affected the killing number of tumor cells. A somewhat similar case was studied by Bokhari, Kara and Zaman [3] in which the Lie-symmetry method was used to obtain some exact solutions for their model. The thesis is organized into five chapters. The chapter 2 describes some basic notions and ideas of Lie-symmetry. In chapter 3, Lie-symmetry analysis of the tumor equation studied by Moyo and Leach [11] is presented. The symmetry algebra is then used to obtain some exact solutions as an illustration. The chapter 4 is devoted to the treatment of the non-linear model arising from case of killing rate depending upon the concentration of tumor cells. In chapter 5 we present the conclusion and point out some directions of future study.

CHAPTER 2

BASIC NOTIONS IN LIE SYMMETRY ANALYSIS

2.1 Introduction

The problem of finding solutions of non-linear differential equations has attracted a lot of attractions as many real life situation give rise to non-linear problems. Many studies have approach such problems through approximate analytical methods such as perturbation method, series expansion methods such as homotopy or homotopy perturbation method or numerical methods. An elegant and in generous method was proposed by Sophus Lie in 1881 usually known as Lie-symmetry method. This method was based upon finding symmetries of the differential equation and using these to find similarity, transformations that would yield differential equations in terms of fewer independent variables or reduce these to linear or simpler ones. This approach has been successfully used in the last two decades to find exact solutions of some challenging non-linear problems. One may refer to [2], [18], for an exposition and many applications of this method. In the following sections, we briefly present some definitions and notions that one essential for the theory forward by Lie.

2.2 Lie Groups

Lie groups are important in mathematical analysis, physics and geometry because they serve to describe the symmetry of analytical structures. Lie groups arise as groups of symmetries of some object, or more precisely, as local groups of transformations acting on some manifolds.

2.3 Groups

Definition 2.1 *A group is a set G together with a binary operation $'*'$ called group operation, satisfying following properties:*

1. **Closure**

   For any two elements α and β of group G there exist an element $\gamma \in G$ such that,

   $$\alpha * \beta = \gamma.$$

2. **Associativity**

   For any three elements α, β and γ in G,

   $$\alpha * (\beta * \gamma) = (\alpha * \beta) * \gamma.$$

3. **Identity Element**

   There exists a unique element e in G such that,

   $$\alpha * e = e * \alpha = \alpha, \qquad \forall \alpha \in G.$$

4. **Inverse Element**

   For each element α in G there exists a unique element α^{-1} in G such that,

   $$\alpha * \alpha^{-1} = \alpha^{-1} * \alpha = e.$$

Definition 2.2 (Abelian Group) *A group G is said to be Abelian if in addition to above properties it satisfies the property:*

$$\alpha * \beta = \beta * \alpha, \qquad \forall\, \alpha, \beta \in G.$$

Definition 2.3 (Subgroup) *Let H be a subset of G. Then H is said to be a subgroup of G if it satisfies all the conditions of the group $(G, *)$ under the same binary operation $'*'$.*

Example 1

1. *The set $\mathbb{Z}$, of integers is a group under group operation $'+'$. The identity element of the group $(Z, +)$ is 0 and the inverse of each element $\alpha \in \mathbb{Z}$ is $-\alpha$. It is also an abelian group.*

2. *Another example of abelian group is the group $(R,+)$, where $\mathbb{R}$ is set of real numbers, with identity element 0 and the inverse of each element α is $-\alpha$. Since $\mathbb{Z} \subset \mathbb{R}$ therefore the group $(Z,+)$ is a subgroup of $(R,+)$.*

3. *Similarly $(R\backslash 0, x)$ is a group having identity element 1 and the inverse of each element α* is 1 α.

2.4 Groups of Transformations

Definition 2.4 *Let G be a set of transformation and $G_i \in G$ such that,*

$$G_i : \alpha \to \tilde{\alpha}(\alpha; \in),$$

where α and $\tilde{\alpha}$ both belong to the set $S \subset \mathbb{R}^n$ and the parameter $\epsilon \in A \subset \mathbb{R}$ with composition law $\psi(\epsilon, \delta)$ for all, $\epsilon, \delta \in A$ satisfying the conditions:

1. *G_i is a one-to-one transformation for each i and for all $\epsilon \in A$.*

2. *(A,ψ) is a group*

3. *For the identity element e of the group (A,ψ).$\tilde{\alpha} = \alpha$ i.e.*

$$G_i(\alpha; e) = \alpha, \qquad \forall i.$$

4. *Let $\tilde{\alpha} = G_i(\alpha; \epsilon)$ then,*

$$\tilde{\tilde{\alpha}} = G_i(\tilde{\alpha}; \delta) = G_i(\alpha; \psi(\epsilon, \delta)).$$

2.5 Lie Groups of Transformations

Definition 2.5 *A transformation group G with composition law ψ is said to be a Lie group of transformations of one-parameter if:*

1. *The parameter ϵ is continuous. i.e. the set A is an interval in $\mathbb{R}$.*

2. *Each element G_i of the group G is an infinitely differentiable function of $\alpha \in S \subset \mathbb{R}^n$.*

3. *The composition function $\psi(\epsilon, \delta)$ is an analytic function.*

Equivalently, a group of infinitesimal point transformations of one parameter ϵ is a transformation group that is invertible and has an identity transformation. Being an invertible means that repeated application of the transformation leads to the transformation of the same family. Mathematically, this statement can be recast as:

Let $\tilde{x} = x(x, y, ; \epsilon)$ and $\tilde{y} = y(x, y, ; \epsilon)$ be a transformation group such that

1. $\tilde{\tilde{x}} = \tilde{x}(\tilde{x}; \tilde{y}; \tilde{\epsilon}) = \tilde{\tilde{x}}(x, y; \tilde{\tilde{\epsilon}})$ for some $\tilde{\tilde{\epsilon}} = \tilde{\epsilon}(\tilde{\epsilon}, \epsilon)$.

2. There exist ϵ_0 such that,

$$\tilde{x}(x, y; \epsilon_0) = x, \qquad \tilde{y}(x, y; \epsilon_0) = y,$$

then it is called an one-parameter group of point transformations.

Example 2 *The transformation defined by,*

$$G_i(x, y) \to (\tilde{x}, \tilde{y}),$$

such that.

$$\tilde{x} = x \cos \epsilon - y \sin \epsilon, \qquad \tilde{y} = x \sin \epsilon + y \cos \epsilon,$$

where ϵ is an infinitesimal parameter forms the group of **rotations** *transformations.*

Since,

$$\tilde{\tilde{x}} = \tilde{x} \cos \delta - \tilde{y} \sin \delta = x \cos(\epsilon + \delta) - y \sin(\epsilon + \delta),$$

$$\tilde{\tilde{y}} = \tilde{x} \sin \delta + \tilde{y} \cos \delta = x \sin(\epsilon + \delta) + y \cos(\epsilon + \delta).$$

Also for $\epsilon = 0$ we have

$$\tilde{x} = x, \qquad and \qquad \tilde{y} = y.$$

Therefore, the above transformations constitutes a one-parameter group of Lie point transformations, where,

$$\psi(\epsilon, \delta) = \epsilon + \delta.$$

Example 3 *A Group of* **translations** *in the plane is defined as,*

$$\tilde{x} = x + \epsilon, \qquad and \qquad \tilde{y} = y + \epsilon.$$

In this case,

$$\tilde{\tilde{x}} = \tilde{x} + \delta = x + \epsilon + \delta, \qquad and \qquad \tilde{\tilde{y}} = \tilde{y} + \delta = y + \epsilon + \delta,$$

with composition law and identity element given respectively as,

$$\psi(\epsilon, \delta) = \epsilon + \delta, \qquad and \qquad \epsilon_0 = e^{\epsilon} y.$$

Therefore the group of translations is Alie Group.

Example 4 *Consider the* **reflection** *transformations defined as,*

$$\tilde{x} = -x, \qquad \text{and} \qquad \tilde{y} = -y.$$

Since,

$$\tilde{\tilde{x}} = -\tilde{x} = -(-x) = x, \qquad \text{and} \qquad \tilde{\tilde{y}} = -\tilde{y} = -(-y) = y,$$

which shows that it is not invertible hence does not form a Lie group of transformation.

2.6 Infinitesimal Transformations

Consider one parameter (ϵ) Lie group of transformation with identity $\epsilon_0 = 0$ and composition law ψ defined as,

$$\tilde{\alpha} = G_i(\alpha; \epsilon). \tag{2.1}$$

Taylor expansion of the transformation (2.1) about $\epsilon_0 = 0$ is given as,

$$\begin{aligned} \tilde{\alpha} &= G_i(\alpha; \epsilon_0) + (\epsilon - \epsilon_0) \left.\frac{\partial G_i(\alpha; \epsilon)}{\partial \epsilon}\right|_{\epsilon=\epsilon_0} + 0(\epsilon^2) \\ &= \alpha + \epsilon \left.\frac{\partial \tilde{\alpha}}{\partial \epsilon}\right|_{\epsilon=0} + 0(\epsilon^2), \end{aligned} \tag{2.2}$$

where $\left.\frac{\partial \tilde{\alpha}}{\partial \epsilon}\right|_{\epsilon=0} = \xi^{\alpha}(\alpha)$.

In particular for $(x, y) \in \mathbb{R}^2$ the Taylor expansion of transformation G_1 such that,

$$G_1 : (x, y) \rightarrow (\tilde{x}, \tilde{y}),$$

is given as,

$$\begin{aligned} \tilde{x} &= x + \epsilon \left.\frac{\partial \tilde{x}}{\partial \epsilon}\right|_{\epsilon=0} + \ldots, \\ \tilde{y} &= y + \epsilon \left.\frac{\partial \tilde{y}}{\partial \epsilon}\right|_{\epsilon=0} + \ldots \end{aligned} \tag{2.3}$$

Substituting,

$$\left.\frac{\partial \tilde{x}}{\partial \epsilon}\right|_{\epsilon=0} = \xi(x,y), \qquad \textit{and} \qquad \left.\frac{\partial \tilde{y}}{\partial \epsilon}\right|_{\epsilon=0} = \eta(x,y).$$

in (2.3) reduces it to,

$$\begin{aligned} \tilde{x} &= x + \epsilon\, \xi(x,y) + \ldots, \\ \tilde{y} &= y + \epsilon\, \eta(x,y) + \ldots \end{aligned}$$

This is called the *Infinitesimal Transformation* and the components $\xi(x,y)$ and $\eta(x,y)$ are called ***infinitesimals*** of the transformation. Transformation (2.1) can be found from the component $\xi(\alpha)$ by integrating

$$\frac{\partial \tilde{\alpha}}{\partial \epsilon} = \xi(\tilde{\alpha}). \tag{2.4}$$

with initial condition $\tilde{\alpha}|_{\epsilon=0} = G_i|_{\epsilon=0} = \alpha$.

Theorem 2.1 **(First Fundamental Theorem of Lie)** *There exists a parametrization $\tau(\epsilon)$ such that the Lie group of transformations $\tilde{\alpha} = G_i(\alpha;\epsilon)$ is equivalent to the solution of the initial value problem for the system of first order differential equations,*

$$\frac{\partial \tilde{x}}{d\tau} = \xi(\tilde{x}). \tag{2.5}$$

with,

$$\tilde{x} = x \qquad \textit{when} \qquad \tau = 0. \tag{2.6}$$

2.7 Infinitesimal Generator

Consider the transformation,

$$\tilde{\alpha} = G_i(\alpha, \epsilon), \tag{2.7}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n) \in \mathbb{R}^n$. Then the operator defined by,

$$\chi = \xi(\alpha) \cdot \nabla = \sum_{k=1}^{n} \xi^k(\alpha) \frac{\partial}{\partial \alpha_k}, \tag{2.8}$$

is called an *infinitesimal generator* of the one parameter group of transformation (2.7) where $\xi^k = \frac{\partial \delta_k}{\partial_\epsilon}\Big|_{\epsilon=0}$ give the components of the tangent vector χ_α.

Consider an arbitrary point $(x, y) \in \mathbb{R}^2$ and the transformation given in

$$\begin{aligned} \bar{x} &= x + \epsilon \xi(x, y) + \ldots \\ \bar{y} &= y + \epsilon \eta(x, y) + \ldots \end{aligned}$$

the symmetry generator corresponding to this transformation is,

$$\chi = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y},$$

where $\xi(x, y) = \frac{\partial \bar{x}}{\partial \epsilon}\Big|_{\epsilon=0}$ and $\eta(x, y) = \frac{\partial \bar{y}}{\partial \epsilon}\Big|_{\epsilon=0}$.

Any transformation (2.7) can be determined completely with the help of infinitesimal generator ξ by integrating,

$$\xi^k(\tilde{\alpha}) = \frac{\partial \tilde{\alpha}_k}{\partial \epsilon}, \tag{2.9}$$

with initial condition $\tilde{\alpha}_k\big|_{\epsilon=0} = \alpha_k$.

Theorem 2.2. *The one-parameter Lie group of transformations $\tilde{\alpha} = G_i(\alpha; \epsilon)$ is* *equivalent to:*

$$\begin{aligned}
\tilde{\alpha} &= e^{\epsilon\chi}\alpha \\
&= \alpha + \epsilon\chi\alpha + \frac{\epsilon^2}{2}\chi^2\alpha + \dots \\
&= \left[1 + \epsilon\chi + \frac{\epsilon^2}{2}\chi^2 + \dots\right]\alpha \\
&= \sum_{k=0}^{\infty}\frac{\epsilon^k}{k!}\chi^k\alpha,
\end{aligned} \tag{2.10}$$

where the operator χ is defined by (2.8).

Example 5. *Consider the group of rotations defined as,*

$$\tilde{x} = x\cos\epsilon - y\sin\epsilon, \qquad \tilde{y} = x\sin\epsilon + y\cos\epsilon.$$

The components of symmetry generator are,

$$\xi(x,y) = \left.\frac{\partial\tilde{x}}{\partial\epsilon}\right|_{\epsilon=0} = -y, \qquad \textit{and} \qquad \eta(x,y) = \left.\frac{\partial\tilde{y}}{\partial\epsilon}\right|_{\epsilon=0} = x. \tag{2.11}$$

The inverse problem is to find the corresponding transformation of a symmetry generator.

2.8 Lie Algebras

Lie Algebra is a vector space, equipped with bilinear product $[.,.] : V \times V \to V$ satisfying (for all vector fields X_i, X_j, X_k belonging to V) with the following properties:

1. $[X_i, X_j] = 0 \quad i = j$

2. $[X_i, X_j] = -[X_j, X_i]$,

3. Any three infinitesimal symmetry generators X_i, X_j and X_k, satisfy the Jacobi's identity,

$$[X_i, [X_j, X_k]] + [X_k, [X_i, X_j]] + [X_j, [X_k, X_i]] = 0.$$

where the commutator operator $[,]$ for any two symmetry generators X_i, X_j is defined, as in by

$$[X_i, X_j] = X_i X_j - X_j X_i. \tag{2.12}$$

Definition 2.4

Let G be an r-parameter Lie group of transformations with basis $\{X_1, X_2, \ldots, X_r\}$, where X_i is an infinitesimal symmetry generator corresponding to the parameter ϵ_i. Then the Lie group G of transformations forms an 4-dimensional Lie algebra G^r over the field $F = \mathbb{R}$ with respect to commutation law.

Example 6

The group of rigid motions in $\mathbb{R}^2$ is the three-parameter Lie group of transformations of rotations and translations in $\mathbb{R}^2$ given by

$$\begin{aligned} \tilde{x} &= x\cos\epsilon_1 - y\sin\epsilon_1 + \epsilon_2 \\ \tilde{y} &= x\sin\epsilon_1 + y\cos\epsilon_1 + \epsilon_3 \end{aligned} \tag{2.13}$$

The corresponding infinitesimal generators are given by

$$\begin{aligned} X_1 &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \\ X_2 &= \frac{\partial}{\partial x}, \\ X_3 &= \frac{\partial}{\partial y}. \end{aligned} \tag{2.14}$$

The commutator table of the above Lie point symmetries is as follows:

$[X_i, X_j]$	X_1	X_2	X_3
X_1	0	$-X_3$	X_2
X_2	X_3	0	0
X_3	$-X_2$	0	0

Table 1: Commutator Table

Example 2 The similitude group in $\mathbb{R}^2$ consists of uniform scalings and rigid motions in $\mathbb{R}^2$. It is the four-parameter Lie group of transformations given by

$$\begin{aligned} \tilde{x} &= e^{\epsilon_4}(x \cos \epsilon_1 - y \sin \epsilon_1) + \epsilon_2 \\ \tilde{y} &= e^{\epsilon_4}(x \sin \epsilon_1 + y \sin \epsilon_1) + \epsilon_3 \end{aligned} \tag{2.15}$$

The corresponding relation are given as table:

$[X_i, X_j]$	X_1	X_2	X_3	X_4
X_1	0	$-X_3$	X_2	0
X_2	X_3	0	0	X_2
X_3	$-X_2$	0	0	X_3
X_4	0	$-X_2$	$-X_3$	0

Table 2: Commutator Table

Definition 2.5

A subset A of Lie algebra G is called a subalgebra of G if it is closed under the commutation operator, i.e for all $X_\alpha, X_\beta \in A$, $[X_\alpha, X_\beta] \in A$.

2.9 Solvable Lie Algebras

The order of an n^{th} order ordinary differential equation (ODE) can be reduced constructively by two if it admits of Lie algebra of transformations of two parameters. But for an r-parameter Lie algebra ($r \geq 3$) the order of the differential equation can be reduced constructively by p, if there exist a p-dimensional solvable subalgebra.

Definition 2.6

A subalgebra $A \subset G$ is called an ideal or normal subalgebra of G if $[g, \alpha] \in A$ for all $a \in A, g \in G$.

Definition 2.7

A^p is p-dimensional solvable Lie algebra if there exists a chain of subalgebras, $A^1 \subset A^2 \subset A^3 \subset \ldots \subset A^{p-1} \subset A^p$ such that A^{i-1} is an ideal of A^i for all $i = 2, 3, \ldots, p$.

Definition 2.8

An algebra G is called an abelian Lie algebra if $[X_\alpha, X_\beta] = 0$ for all $X_\alpha, X_\beta \in G$.

Theorem 2.2

Every two-dimensional Lie algebra and every Abelian Lie algebra is a solvable Lie algebra.

2.10 Structure Constants

Theorem 2.3: (Second fundamental Theorem of Lie)

The commutator of any two infinitesimal generator of an r-parameter Lie group of transformations is also an infinitesimal generator. In particular,

$$[X_\alpha, X_\beta] = \sum_{\gamma=1}^{r} C_{\alpha\beta}^{\gamma} X_\gamma \in G \tag{2.16}$$

where $C_{\alpha\beta}^{\gamma}$ are the structure constants.

Definition 2.9 (Communication Relations)

For an r-parameter Lie group of transformations with basis $X_1, X_2, \ldots, X_r$ the relations defined by equation (2.16) are called commutation relations.

Definition 2.4 (Third Fundamental Theorem of Lie)

The structure constants, defined by commutation (2.16), satisfy the relations:

1. $C^{\gamma}_{\alpha\beta} = -C^{\gamma}_{\beta\alpha}$ **(skew symmetry).**

2. $C^{\rho}_{\alpha\beta}C^{\delta}_{\rho\gamma} + C^{\rho}_{\beta\gamma}C^{\delta}_{\rho\alpha} + C^{\rho}_{\gamma\alpha}C^{\delta}_{\rho\beta} = 0$ **(Jacobi identity).**

2.11 Prolongation

In order to apply the transformations (2.3) to an n^{th} order partial differential equation (PDE), one needs to extend the infinitesimal symmetry generator (2.7) to include all derivatives of the dependent variables. In this section we discuss the prolongation formula for a PDE which consists of $'p'$ dependent and $'q'$ independent variables. Since we will be dealing with a PDE of order two, we later deduce a prolongation formula for a second order PDE in which there is only one dependent variable.

Let

$$F(x; u, u^{(1)}, u^{(2)}, \ldots, u^{(n)}) = 0. \tag{2.17}$$

be an n^{th} order PDE with q independent variables $x = (x_1, x_2, x_3, \ldots, x_q), p$ dependent variables $u = (u^1, u^2, \ldots, u^p)$ and the derivatives of dependent up to order n. In this case the infinitesimal symmetry generator associated with this equation becomes

$$X = \sum_{i=1}^{q} \xi_i(x, u)\frac{\partial}{\partial x_i} + \sum_{k=1}^{p} \phi^k(x, u)\frac{\partial}{\partial u^k}. \tag{2.18}$$

Then the prolongation of the generator (2.18) is obtained by extending it to include all the derivatives as

$$X^p = X + \sum_{k=2}^{p}\sum_{j} \phi^{jk}(x, u, u^{(n)})\frac{\partial}{\partial u_{jk}}. \tag{2.19}$$

where ϕ^j and ϕ^{jk} are given by

$$\phi^j = D_j(\phi - \xi^i u_{,i}) + \xi^i u_{,ji} \tag{2.20}$$

$$\phi^{jk} = D_k D_j(\phi - \xi^i u_{,i}) + \xi^i u_{,jki} \tag{2.21}$$

in which D_i represents the total derivative given by the formula

$$D_i = \frac{\partial}{\partial x_i} + \sum_j u_{j,i} \frac{\partial}{\partial u_j}. \tag{2.22}$$

2.12 Invariance

A Lie group of transformations can have invariant functions, surfaces, curves, and invariant points. The invariance can transform the complicated nonlinear conditions into simpler linear conditions under the corresponding infinitesimal generator of the symmetry group. The symmetry group of the system transforms its solutions to other solutions giving new invariant solutions of the system.

2.12.1 Invariance of a function

Definition 2.10

Let $\bar{x} = G_i(x, \epsilon)$ be the Lie group of transformations of one parameter ϵ and let $f(u)$ be an infinitely differentiable function. The function $f(u)$ is said to be an invariant function if and only if

$$f(\bar{x}) = f(x) \tag{2.23}$$

Theorem 2.6

Given Lie group of transformation $\bar{x} = G_i(x, \epsilon)$ with symmetry generator X, the identity,

$$f(\bar{x}) = f(x) + \epsilon \tag{2.24}$$

holds if

$$Xf(x) = 1 \tag{2.25}$$

and conversely.

2.12.2 Invariance of surface

Definition 2.11

A surface $f(x) = 0$ is an invariant surface for the Lie group of symmetry transformation of one parameter ϵ if and only if

$$f(\bar{x}) = 0 \qquad when \qquad f(x) = 0 \tag{2.26}$$

Theorem 2.7.

Let $f(x) = 0$ to be a surface and let $\bar{x} = G_i(x, \epsilon)$ be a one-parameter Lie group of transformations. The surface $f(x) = 0$ is said to be an invariant surface under the symmetry transformation if and only if

$$Xf(\bar{x}) = 0 \qquad when \qquad f(x) = 0 \tag{2.27}$$

2.12.3 Invariance of a Partial Differential Equation

Consider a system of partial differential equation of order n with q independent $x = (x_1, x_2, \ldots, x_q)$ and p dependent variable $u = (u^1, u^2, \ldots, u^p)$, given by

$$F_\mu(x, u, \partial u, \partial^2 u, \ldots, \partial^n u) = 0 \tag{2.28}$$

where

$$\mu = 1, 2, 3, \ldots, k$$

The derivative of order m is denoted as,

$$u_j^\alpha = \frac{\partial^m u^\alpha}{\partial x_{j1} \partial x_{j2} \ldots \partial x_{jm}} \tag{2.29}$$

where $1 \leq j_i \leq p$ for all $i = 1, 2, \ldots, m$ and the order of m-tuple of integers $j = (j_1, j_2, \ldots, j_m)$ indicates the order of the derivative.

Theorem 2.8 (Invariance Criterion of PDEs)

The PDE (2.17) is said to be invariant under the symmetry generator (2.19) when

$$X^{p}(F(x;u,u^{(1)},u^{(n)}))|_{Eq(2.17)=0}=0. \tag{2.30}$$

Example 8 (Diffusion Equation)

We demonstrate how a PDE remains invariant under a group of transformation. Consider the diffusion equation

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2} \tag{2.31}$$

and the transformation

$$\bar{x}=e^{\alpha}x \tag{2.32}$$

$$\bar{t}=e^{\alpha}t \tag{2.33}$$

$$\bar{u}=e^{\alpha}u \tag{2.34}$$

Different values of a,b,c give different elements of the group of transformations ($a=b=c=0$) gives the identity transformation. It is straightforward to see

$$\frac{\partial\bar{u}}{\partial\bar{t}}=\frac{\partial}{\partial\bar{t}}(e^{\epsilon}u)=\frac{\partial}{\partial t}(e^{\epsilon}u)\frac{\partial t}{\partial\bar{t}}=e^{c-b}\frac{\partial u}{\partial t} \tag{2.35}$$

$$\frac{\partial^2\bar{u}}{\partial\bar{x}^2}=e^{c-2a}\frac{\partial^2 u}{\partial x^2}. \tag{2.36}$$

The diffusion equation transforms into

$$\frac{\partial\bar{u}}{\partial\bar{t}}-\frac{\partial^2\bar{u}}{\partial\bar{x}^2}=e^{c-2a}\left(\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}\right)=\frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2}. \tag{2.37}$$

2.12.4 Invariance of surface

The infinitesimal transformations and the infinitesimal symmetry generators of the Lie group of a partial differential equation can be calculated by a systematic computational procedure in the light of Theorem (2.8) and used the prolongation formula (2.19). The first step is to find the one-parameter symmetry generator X. The coefficients $\xi^i(x, u)$ and $\phi_\alpha(x, u)$ of the symmetry generator X will be functions of x, u. The symmetry generator X will be prolonged to the order equivalent to the order of partial differential equation.

Application of a prolonged symmetry generator to the partial differential equation using the theorem (2.8) of the infinitesimal criterion for the invariance of PDE gives a general equation that involves x, u and the derivatives of u with respect to x, as well as, $\xi^i(x, u), \phi_\alpha(x, u)$ and their partial derivatives with respect to x and u. By comparing the coefficients of the partial derivatives of u we get a system of equations known as determining equations for the coefficients functions $\xi^i(x, u)$ and $\phi_\alpha(x, u)$. The general solution of this system of determining equations determines the most general expressions for $\xi^i(x, u)$ and $\phi_\alpha(x, u)$; thus giving the general infinitesimal symmetry generator X.

CHAPTER 3

Tumor Equation with Variable Profile

3.1 Introduction

In this chapter we shall investigate the tumor model in which the killing rate depends upon the space and time variables. The equation is

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} - k(x,t)u = 0, \tag{3.1}$$

has been investigated by Moyo and Leach [11] who have given some exact solutions in this case. Our objective here is to perform Lie symmetry analysis of his equation and present some exact solutions for some interesting cases. One may note that the tumor equation (1.3) in spherical coordinates with complete radial symmetry reduces to the equation (3.1) by using the substitutions $u(x,t) = rn(r,t)$ and $x = r$ as mentioned in chapter 1.

3.2 Lie Symmetry Analysis

Consider the infinitesimal generator [2, 18]

$$X = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u} \tag{3.2}$$

Since our equation is of order 2, we prolong the above generator as follows:

$$X_p = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u} + \phi^x\frac{\partial}{\partial u_x} + \phi^t\frac{\partial}{\partial u_t} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xt}\frac{\partial}{\partial u_{xt}} + \phi^{tt}\frac{\partial}{\partial u_{tt}} \quad (3.3)$$

In the above expression, the coefficients $\phi^x, \phi^t, \phi^{xx}, \phi^{xt}$ and ϕ^{tt} of the prolonged generator are functions of (x, t, u) and can be determined by

$$\begin{aligned} \phi^i &= D_i(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,i} + \eta u_{y,i} + \tau u_{t,i}, \\ \phi^{ij} &= D_i D_j(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,i} + \eta u_{y,ij} + \tau u_{t,ij}, \end{aligned}$$

where D_i represents total derivative and subscripts of u partial derivative with respect to the respective coordinates. Note that in this notation x_1 corresponds to x and x_2 to t. Now, by using the above formulas we have

$$\phi^x = \phi_x + \phi_u u_x - u_t(\tau_x + u_x\tau_u) - u_x(\xi_x + \xi_u u_x) \quad (3.4)$$

$$\phi^t = \phi_t + \phi_u u_t - u_t(\tau_t + u_t\tau_u) - u_x(\xi_t + \xi_u u_t)$$

$$\phi^{xx} = \phi_{xx} + 2\phi_{xu}u_x + \phi_{uu}u_{xx} - 2u_{tx}(\tau_x + \tau_u u_x) -$$

$$u_t(\tau_{xx} + 2\tau_{xu}u_x + \tau_{uu}u_x^2 + \tau_u u_{xx}) - 2u_{xx}(\xi_x + \xi_u u_x) -$$

$$u_x(\xi_{xx} + 2\xi_{xu}u_x + \xi_{uu}u_x^2 + \xi_u u_{xx})$$

By using (3.4) in our equation (3.4) we obtain

$$X_p\{PDE(3.1)\} = 0, \quad (3.5)$$

when PDE (3.1) is satisfied. This gives

$$-u\xi k_x - u\tau k_t - k\phi - \phi^t + \phi^{xx} = 0. \quad (3.6)$$

After putting $u_t = u_{xx} - k(x,t)u$, we obtain

$$-uk_x\xi - uk_t\tau - k\phi - \phi_t - \phi_u(u_{xx} - ku) + (u_{xx} - ku)\tau_t +$$

$$\left(u_{xx}^2 + k^2u^2 - 2kuu_{xx}\right)\tau_u + u_x\xi_t + u_x\left(u_{xx} - k_u\right)\xi_u + \phi_{xx} +$$

$$2\phi_{xu}u_x + \phi_{uu}u_x^2 + \phi_u u_{xx} - 2u_{tx}\tau_x - 2u_{tx}u_x\tau_x - \left(u_{xx} - ku\right)\tau_x -$$

$$\left(u_{xx} - ku\right)\tau_{xx} - 2\left(u_{xx} - ku\right)u_x\tau_{xu} - u_x^2\left(u_{xx} - ku\right)\tau_{uu} - \left(u_{xx} - ku\right)u_{xx}\tau_u -$$

$$2u_{xx}\xi_x - 3u_{xx}u_x\xi_u - u_x\xi_{xx} - 2u_x^2\xi_{xu} - u_x^3\xi_{uu} = 0$$

Then by comparing the coefficients of derivatives of u of both sides we get the following equations:

$$u_{xx}u_x : -3\xi_u + \xi_u - 2\tau_{xu} = 0 \tag{3.7}$$

$$u_{xx} : -2\xi_x - \tau_{xx} + \tau_t + ku\tau_u + \phi_u = 0$$

$$u_{xx}u_x^2 : -\tau_{uu} = 0$$

$$u_{tx}u_x : \tau_x = 0$$

$$u_x^2 : ku\tau_{uu} + \phi_{uu} - 2\xi_{uu} - 2\xi_{xu} = 0$$

$$u_t^2 : \tau_u = 0$$

$$u_x : 2ku\tau_{xu} + 2\phi_{xu} - ku\xi_u + \xi_t - \xi_{xx} = 0$$

$$u_{xt} : -2\tau_x = 0$$

$$u_x^3 : \xi_{uu} = 0$$

$$1 : -uk_x\xi - uk_t\tau - k\phi - \phi_t + ku\phi_u - ku\tau_t + \phi_{xx} + k^2u^2\tau_u + ku\tau_{xx} = 0$$

So, the reduced system can be given as follows:

$$\tau = \tau(t), \tag{3.8}$$

$$\xi = \xi(x, t), \tag{3.9}$$

$$\phi = \phi_1(x, t)u + \phi_2(x, t), \tag{3.10}$$

$$\xi_t - \xi_{xx} + 2\phi_{1x} = 0, \tag{3.11}$$

$$\tau_t - 2\xi_x = 0, \tag{3.12}$$

$$k_x\xi + k_t\tau + \phi_{1t} + k\tau_t - \phi_{2xx} = 0. \tag{3.13}$$

$$k\phi_2 + \phi_{2t} - \phi_{2_{xx}} = 0. \tag{3.14}$$

The solutions of the above equations depend on the value of $k(x, t)$. Thus we study some cases for typical values of $k(x, t)$,

Case(i):

In this case of no or ineffective treatment of the Tumor equation. Thus $k(x) = 0$ leads to the heat equation

$$u_t = u_{xx}. \tag{3.15}$$

Lie symmetries admitted by the PDE (3.15) have infinitesimal generators as

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial x}, \\
X_2 &= \frac{\partial}{\partial t}, \\
X_3 &= x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, \\
X_4 &= xt\frac{\partial}{\partial x} + t^2\frac{\partial}{\partial t} - \left[\frac{x^2}{4} + \frac{t}{2}\right] u\frac{\partial}{\partial u}, \\
X_5 &= t\frac{\partial}{\partial x} - \frac{x}{2}u\frac{\partial}{\partial u}, \\
X_6 &= u\frac{\partial}{\partial u}.
\end{aligned}$$

The corresponding relations between the infinitesimal generators of the heat equation are given in Table (3-1).

Table 3.1: Commutator table of the heat equation

$[X_\alpha, X_\beta]$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	X_1	X_5	0	0
X_2	0	0	X_2	X_3	X_1	0
X_3	$-X_1$	$-X_2$	0	$2X_4$	X_5	0
X_4	$-X_5$	$-X_3$	$-2X_4$	0	0	0
X_5	0	$-X_1$	$-X_5$	0	0	0
X_6	0	0	0	0	0	0

We now demonstrate the use of some of these symmetry generators to reduce the PDE (3.15)

As a first step we pick a generator, say:

$$X_2 = \frac{\partial}{\partial t}$$

The characteristic equations are

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0} \tag{3.17}$$

The first two terms gives

$$\frac{dx}{0} = \frac{dt}{1} \quad i.e. \quad x = \alpha. \tag{3.18}$$

Last two terms gives

$$\frac{dt}{1} = \frac{du}{0} \quad i.e. \quad u = w(\alpha). \tag{3.19}$$

Now, we re-cast equation (3.15) into these new variables

$$u_t = w_t = 0 \tag{3.20}$$

$$u_x = w_\alpha. \tag{3.21}$$

$$u_{xx} = w_{xx} = \frac{\partial w_x}{\partial \alpha}\frac{\partial \alpha}{\partial x} = w_{\alpha\alpha}. \tag{3.22}$$

Thus the equation (3.15) reduces to

$$w_{\alpha\alpha} = 0 \tag{3.23}$$

This has the solutions

$$w = c_1 + c_2\alpha \tag{3.24}$$

for some constants c_1, c_2.

We notice that this gives

$$u(x,t) = c_1 + c_2 x \tag{3.25}$$

in the original coordinate (x, t).

We can similarly consider the generator

$X_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}$.

The characteristic equations in this case is

$$\frac{dx}{x} = \frac{dt}{2t} = \frac{du}{0} \tag{3.26}$$

This gives $du = 0$ and $u = w(\alpha)$.

Using the above relation one can obtain

$$x^3\alpha^3 w\alpha\alpha + 2x^3\alpha^2 w_\alpha - x^3 w_\alpha = 0, \tag{3.27}$$

i.e.

$$\alpha^3 w\alpha\alpha + (2\alpha^2 - 1)w_\alpha = 0, \tag{3.28}$$

put $w_\alpha = p(\alpha)$ to get

$$p_\alpha + \left(\frac{2\alpha^2 - 1}{\alpha^3}\right) p = 0, \tag{3.29}$$

which is a linear first ordinary differential equation.

Case(ii):

when $\phi_1 = \phi_1(t)$ and $\phi_2 = \phi_2(t)$:

We get from (3.14) that

$$k\phi_2 + \phi_2' = 0 \tag{3.30}$$

This means that

$$k = k(t) \tag{3.31}$$

From (3.30) we find that

$$\frac{\phi_2'}{\phi_2} = -k \tag{3.32}$$

this means that

$$\phi_2 = Ae^{-\int kdt} \tag{3.33}$$

Where A is a constant. Also from (3.13) we get

$$k'\tau + \phi_1' + k\tau' = 0 \tag{3.34}$$

This means that

$$(k\tau + \phi_1)' = 0 \Rightarrow k\tau + \phi_1 = c_0 \tag{3.35}$$

So

$$\phi_1 = c_0 - k\tau \tag{3.36}$$

where c_0 is a constant. From (3.11) and (312) we get $\xi_t = 0 \Rightarrow \xi = \xi(x)$

From (3.12) we have $\tau_t = 2\xi_x \Rightarrow \tau_{tt} = 0$ So

$$\tau = c_1 t + c_2 \tag{3.37}$$

where c_1, c_2 are constants. Then from above equations we have $2\xi_x = c_1$

So

$$\xi = \frac{1}{2}c_1 x + c_3 \tag{3.38}$$

for arbitrary constant c_3. Finally we can re-write the solution of the determining equations in this case as follow:

$$\begin{aligned} \xi &= \frac{1}{2}c_1 x + c_3 \\ \tau &= c_1 t + c_2 \\ \phi &= \left[c_0 - k\left(c_1 t + c_2\right)\right] u + Ae^{\int k dt}. \end{aligned} \tag{3.39}$$

where c_0, c_1, c_2, c_3, A are constants.

Case(iii):

Let us assume the form of the killing function as

$$k(x) = \frac{\mu}{x^2}$$

where μ is a constant.

The symmetry algebra is given by

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t}, \\
X_2 &= 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}, \\
X_3 &= 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} - \left(x^2 - 2t\right) u\frac{\partial}{\partial u}, \\
X_4 &= u\frac{\partial}{\partial u}, \\
X_f &= f(x,t)\frac{\partial}{\partial u}, \qquad (3.40)
\end{aligned}$$

$f(x,t)$ is a solution of PDE (3.1). The commutator table of the above generators is given in Table: (3.2).

Table 3.2: Commutator table $k(x) = \frac{\mu}{x^2}$

$[X_\alpha, X_\beta]$	X_1	X_2	X_3	X_4
X_1	0	$2X_1$	$4X_2 + 2X_4$	0
X_2	$-2X_1$	0	$2X_3$	0
X_3	$-4X_2 - 2X_4$	$-2X_3$	0	0
X_4	0	0	0	0

Let us first use the symmetry generator

$$X_1 = \frac{\partial}{\partial t}.$$

This corresponds to the steady state solutions. The characteristic equations are

$$\frac{dt}{1} = \frac{dx}{0} = \frac{du}{0} \qquad (3.42)$$

i.e. $x = \alpha, \quad u = w(\alpha)$

The PDE (3.41) gives

$$w_{\alpha\alpha} - \frac{\mu}{\alpha^2}w = 0. \qquad (3.43)$$

which is the Cauchy Riemann equation and can be solved as follows.

The auxiliary equation has roots

$$m = \frac{1 \pm \sqrt{1+4\mu}}{2} = m_1, m_2 \tag{3.44}$$

This gives using $\alpha = x$ and $w = u$

$$u = ax^{m_1} + bx^{m_2} \tag{3.45}$$

where m_1, m_2 are given in the equation above.

For $\mu = 2$ as example, this gives

$$u = \frac{a}{x} + bx^2 \tag{3.46}$$

If we use $X_2 = 2t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x}$, a procedure similar to above leads to

$$\alpha = \frac{x}{\sqrt{t}} \text{ and } u = w(\alpha).$$

The PDE (3.41) then reduces to ODE

$$\alpha^2 w'' + 2\alpha^3 w' - \mu f = 0. \tag{3.47}$$

CHAPTER 4

Non-linear Model of Tumor Monitoring

4.1 Introduction

The model of tumor equation growth and modeling discussed in chapter (3) was based upon the killing rate to be dependent on radial distance from the center from the time. In this chapter we consider a more general case in which the killing rate is assumed to be dependent upon the population of the cancer cells. This assumption is based upon the observation that growth or decay is cell population is dependent upon the number of cells present. Discussion of some of the models used in the study of tumor models can be found in [16], [5], [4].

As described in chapter (2) we shall again perform Lie symmetry analysis of the PDE describing this model and obtain exact solutions in some cases.

The governing equation for tumor equation can be written as

$$u_{xx} - \frac{2}{x}u_x - k(u)u = u_t \tag{4.1}$$

where u denotes the number of cancer cells, x the radial distance and $k(u)$ the killing rate.

4.2 Lie Symmetry Analysis

The symmetry generator is given by

$$X = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u} \tag{4.2}$$

Since our equation is of order 2, we then prolong the above generator to be in this form

$$X_p = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial u} + \phi^x\frac{\partial}{\partial u_x} + \phi^t\frac{\partial}{\partial u_t} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xt}\frac{\partial}{\partial u_{xt}} + \phi^{tt}\frac{\partial}{\partial u_{tt}} \tag{4.3}$$

In the above expression, the coefficients of the prolonged generator are functions of (x, t, u) and can be determined by

$$\begin{aligned}
\phi^i &= D_i(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,i} + \eta u_{y,i} + \tau u_{t,i}, \\
\phi^{ij} &= D_i D_j(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,ij} + \eta u_{y,ij} + \tau u_{t,ij},
\end{aligned}$$

where D_i represents total derivative and subscripts of u partial derivative with respect to the respective coordinates.

Now, by using the above formulas in the above prolonged generator we have

$$\frac{1}{x^2}(-3u_x\,\xi_u x^2 k(u)u - u_t\tau_u x^2 k(u)u - 2u_t u_x\tau_{xu}x^2 -$$

$$u_t u_x^2\tau_{uu}x^2 - 2u_{xt}x^2 - 2u_x u_t\xi_u x^2 - \phi x^2 k_u k_u u +$$

$$\phi_u x^2 k(u)u - 2\xi_x x^2 k(u)u + \phi_{xx}x^2 - \phi_t x^2 - \phi_t x^2 - 2x\phi_x + 2\xi u_x +$$

$$+2u_x\phi_{xu}x^2 - u_x\xi_{xx}x^2 - 2u_x^2\xi_{xu}x^2 - u_t\tau_{xx}x^2 + u_x^2\phi_{uu}x^2 -$$

$$u_x^2\xi_{uu}x^2 - 2u_{xt}\tau_x x^2 + u_x\xi_t x^2 + u_t\tau_t x^2 - \phi x^2 k(u) -$$

$$2xu_x\xi_x - 3xu_x^2\xi_u + 2xu_t\tau_x - 2\xi_x x^2 u_t) = 0$$

Following the proceeder outlined in chapter 3 we obtain the following determining equations as follows:

$$-2\xi_{xu}x + \phi_{uu}x - 4\xi_u = 0 \tag{4.4}$$

$$\tau_u xk(u)u - \tau_{xx}x + \tau_t x + 2t_x - 2x\xi_x = 0 \tag{4.5}$$

$$-3\xi_u x^2 k(u)u + 2\xi + 2\phi_{xu}x^2 - \xi_{xx}x^2 + \xi_t x^2 - 2x\xi_x = 0 \tag{4.6}$$

$$-\phi xk(u)u + \phi_u xk(u)u - 2\xi_x xk(u)u + \phi_{xx}x - \phi_t - 2\phi_x - \phi_x k(u) = 0 \tag{4.7}$$

$$\tau_x = 0 \tag{4.8}$$

$$\tau_u = 0 \tag{4.9}$$

$$\tau_{uu} = 0 \tag{4.10}$$

$$\xi_{uu} = 0 \tag{4.11}$$

$$-2\tau_{xu} - 2\xi_u = 0 \tag{4.12}$$

We obtain the following as result of this over determined system.

$$\tau_x = 0 \tag{4.13}$$

$$\tau_u = 0 \tag{4.14}$$

$$\xi_u = 0 \tag{4.15}$$

$$\phi_{uu} = 0 \tag{4.16}$$

$$2\xi + x^2\xi_t + 2x\xi_x - 2x\tau_t - x^2\xi_{xx} + 2x^2\phi_{xu} = 0 \tag{4.17}$$

$$-2ux\phi + x\phi_t + u^2x\phi_u + 2\phi_x - x\phi_{xx} - u^2x\tau_t = 0 \tag{4.18}$$

$$-2\xi_x + \tau_t = 0 \tag{4.19}$$

For $k(u) = k$ we find only one symmetry namely

$$X_1 = \frac{\partial}{\partial t}.$$

As this not interested to us. For our model of tumor monitoring we consider some special cases:

Case(i): $k(u) = e^u$

In this case the determining equations again lead to the minimal symmetry

$$X_1 = \frac{\partial}{\partial t}.$$

Case(ii): $k(u) = \frac{1}{u}$

In this case the following symmetry generators can be found

$$\begin{aligned}
X_1 &= \frac{\partial}{\partial t} \\
X_2 &= x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} \\
X_3 &= t^2\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - \frac{x^2 u}{4}\frac{\partial}{\partial u} \\
X_4 &= -\frac{1}{2}\frac{\partial}{\partial x} + \frac{u}{2x}\frac{\partial}{\partial u} \\
X_5 &= u\frac{\partial}{\partial u} \\
X_6 &= -2t\frac{\partial}{\partial x} + \left(xu + 2\frac{ut}{x}\right)\frac{\partial}{\partial u}
\end{aligned}$$

Case(iii): $k(u) = u$

The determining equations yield

$$\tau_x = 0 \tag{4.20}$$

$$\tau_u = 0 \tag{4.21}$$

$$\xi_{uu} = 0 \tag{4.22}$$

$$\tau_{uu} = 0 \tag{4.23}$$

$$-2\tau_{xu} = 0 \tag{4.24}$$

$$2\xi_{xu} + \phi_{uu} - 4\frac{\xi_u}{x} = 0 \tag{4.25}$$

$$-\tau_u u^2 - \tau_{xx} + \tau_t + 2\frac{\tau_x}{x} - 2\xi_x = 0 \tag{4.26}$$

$$-3\xi_u u^2 + 3\frac{\xi}{x^2} + 2\phi_{xu} - \xi_{xx} + \xi_t - 2\frac{\xi_x}{x} - 2\frac{\tau_t}{x} = 0 \tag{4.27}$$

$$-2\phi u + \phi_u u^2 - 2\xi_x u^2 + \phi_{xx} - \phi_t - 2\frac{\phi_x}{x} = 0 \tag{4.28}$$

The solutions to the above

$$\phi = -u - c_1 \tag{4.29}$$

$$\xi = \frac{1}{2}c_1 x \tag{4.30}$$

$$\tau = c_1 t + c_2 \tag{4.31}$$

Then

$$\begin{aligned} X_1 &= \frac{\partial}{\partial t} \\ X_2 &= -u\frac{\partial}{\partial u} + \frac{x}{2}\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} \end{aligned}$$

Reduction in case (iii). We consider the second generator as X_1 of no investigation.

Now consider the symmetry generator

$$X_2 = t\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}$$

Write characteristic equation:

$$\frac{dt}{t} = \frac{2dx}{x} = \frac{du}{u}$$

First two terms give:

$$\frac{dt}{t} = \frac{2dx}{x} \Rightarrow \alpha = \frac{x^2}{t}$$

Last two terms give:

$$lnt = -lnu + lnw \Rightarrow lnt = lnw - lnu \Rightarrow u = \frac{w}{t}.$$

Then we have

$$\begin{aligned}
u_t &= \frac{-x^2 w_\alpha}{t^3} - \frac{w}{t^2} \\
u_x &= \frac{2x}{t^2} w_\alpha \\
u_{xx} &= 4x^2 \frac{w_{\alpha\alpha}}{t^3} + 2\frac{w_\alpha}{t^2}
\end{aligned}$$

where

$$u = \frac{w}{t}, \qquad \alpha = \frac{x^2}{t}$$

Then the equation

$$u_{xx} - \frac{2}{x} u_x - u_t - u^2 = 0$$

being as:

$$4\alpha w_{\alpha\alpha} + (\alpha - 2) w_\alpha - (1 - w) w = 0.$$

This is a nonlinear ordinary differential equation which can be numerically studied.

CHAPTER 5

Conclusion

5.1 Summary

In this thesis we have studied a mathematical model based upon diffusion process of a tumor in brain. We used the Burgess equation with the additional consideration that the response of the treatment may not be constant in x and t. A symmetry analysis for the resulting partial differential equation in spherical coordinates with complete radial symmetry was performed using Lie-symmetries of this equation. Using some infinitesimal symmetry generators we obtained the corresponding similarity variables. These were used to reduce the governing equation into ODEs. We obtained some exact solutions.

Another interesting situation which we dealt with was the case when the killing rate (based upon the treatment of tumor) depends on the concentration of tumor cells. This case gave rise to a non-linear in homogeneous term. We used symmetry analysis to explore different forms of this killing rate and obtained exact solution.

5.2 Recommendation

The tumor model studied in this thesis has various interesting directions to pursue further investigations. As a first step, we can study a spherical tumor without any (radial or azimuthal) symmetry. This will give rise to a (3+1) partial differential equation and finding the Lie-symmetries could be challenging. However, this would help in further our understanding of the treatment of tumors.

As a next step, we can assume that the killing rate is dependent upon space variables. time and concentration of tumor cells. A completely general model could involve formidable work but use of computer algebra may enable us to study some specific cases of interest.

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